SUMMER 2025 READING GROUP ON ERGODIC THEORY

EXERCISE SHEET 1 (SAMY LAHLOU): CRASH COURSE ON MEASURE THEORY, PART I

Throughout, let (X, \mathcal{B}, μ) be a measure space and let $A_n \in \mathcal{B}$.

Exercise 1 (Monotonicity). If $A_1 \subseteq A_2$, then $\mu(A_1) \leq \mu(A_2)$.

Deduce that if μ is finite, then μ is a bounded function. (Are σ -finite measures bounded?)

Exercise 2 (Inclusion-exclusion). For any $A_1, A_2 \in \mathcal{B}$, we have $\mu(A_1 \cup A_2) + \mu(A_1 \cap A_2) = \mu(A_1) + \mu(A_2)$.

Exercise 3 (Continuity \nearrow). If $(A_n)_{n\in\mathbb{N}}$ is increasing, then $\mu(\bigcup_{n\in\mathbb{N}}A_n) = \lim_{n\to\infty}\mu(A_n)$.

Exercise 4 (Continuity \searrow). If $(A_n)_{n\in\mathbb{N}}$ is decreasing and $\mu(A_1) < \infty$, then $\mu(\bigcap_{n\in\mathbb{N}} A_n) = \lim_{n\to\infty} \mu(A_n)$.

Exercise 5. Show that $\lambda(\mathbb{Q}) = 0$. HINT: What is the Lebesgue measure of singletons?

Let P be a property of some points in X. We say that P holds μ -almost everywhere (or μ -almost surely) if $\{x \in X : x \text{ satisfies } P\}$ is μ -conull.

Exercise 6 (Borel-Cantelli Lemmas). Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of μ -measurable sets.

- 1. If $\sum_{n \in \mathbb{N}} \mu(A_n) < \infty$, then μ -almost every $x \in X$ lives in at-most finitely-many A_n .
- 2. (Measure Compactness). If $\mu(X) < \infty$ and there exists $\varepsilon > 0$ such that $\mu(A_n) \ge \varepsilon$ for all $n \in \mathbb{N}$, then at least an ε -measure set of $x \in X$ lives in infinitely-many A_n 's.

For measurable spaces (X_1, \mathcal{B}_1) and (X_2, \mathcal{B}_2) , define $\mathcal{B}_1 \otimes \mathcal{B}_2 := \langle B_1 \times B_2 : B_i \in \mathcal{B}_i \rangle_{\sigma}$.

Exercise 7. Show that if X_i are second-countable topological spaces, then $\mathcal{B}(X_1 \times X_2) = \mathcal{B}(X_1) \otimes \mathcal{B}(X_2)$.

Exercise 8. Let X be a topological space. A *Cantor set* is a subset $C \subseteq X$ homeomorphic to $2^{\mathbb{N}}$.

- 1. Show that the 'middle-thirds Cantor set' $C \subseteq [0,1]$ is a Cantor set as in the above definition. Moreover, show that $\lambda(C) = 0$. HINT: Recall the construction $C = \bigcap_{n \in \mathbb{N}} C_n$ and use continuity.
- 2. Define a Cantor set $C \subseteq [0,1]$ with positive Lebesgue measure. HINT: fatten the standard construction.

A measurable set $A \subseteq X$ is said to be an *atom* if there is no subset $B \subseteq A$ with $0 < \mu(B) < \mu(A)$. For example, singletons $\{x\}$ are atoms under the Dirac measure δ_x . More generally:

Exercise 9. If (X, \mathcal{B}, μ) is a σ -finite measure space, \mathcal{B} is countably generated (i.e., $\mathcal{B} = \langle \mathcal{B}_0 \rangle$ for some countable $\mathcal{B}_0 \subseteq \mathcal{P}(X)$), and separates points (i.e., if $x \neq y$, then there exists $B \in \mathcal{B}$ such that $x \in B \not\ni y$.), then every atom $A \in \mathcal{B}$ is a singleton.

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