SUMMER 2025 READING GROUP ON ERGODIC THEORY

Exercise Sheet 2 (Samy Lahlou): Crash course on Measure Theory, Part II

Throughout, let (X, μ) be a measure space.

Exercise 1. Continuous maps are Borel. Hint: Define a σ-algebra containing open sets in the codomain.

Exercise 2. In separable metric spaces, pointwise limits of μ-measurable functions are μ-measurable, i.e., if (f_n) is a sequence of μ-measurable maps $f_n : X \to Y$ from a measure space (X, μ) to a separable space Y, and $f := \lim_n f_n$ (pointwise), then $f : X \to Y$ is μ-measurable.

Hint: Let $\mathcal{C} := \{B \in \mathcal{B}(Y) : f^{-1}(B) \in \mathrm{Meas}_\mu\}$. Show that $\mathcal{C}$ is a σ-algebra containing all open set in Y, so $\mathcal{C} = \mathcal{B}(Y)$, as desired. For each $U \subseteq Y$ open, use separability to write $U = \bigcup_{n \in \mathbb{N}} B_n$, where each B_n is a ball whose closure is contained in U, and show that $f^{-1}(U) \in \mathrm{Meas}_\mu$.

Exercise 3. If $f_1, f_2 : (X, \mu) \to \mathbb{R}$ are μ-measurable and $g : \mathbb{R}^2 \to \mathbb{R}$ is Borel, then $g(f_1, f_2) : X \to \mathbb{R}$ is also μ-measurable. In particular, $f_1 + f_2$ and $f_1 \cdot f_2$ are μ-measurable.

Exercise 4. If (f_n) is a sequence of μ-measurable functions $f_n : X \to \overline{\mathbb{R}}$, then $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$, and $\liminf_n f_n$ are also μ-measurable.

Exercise 5. Let $f, g \in L^p(X, \mu)$. If $f \leq g$, then $\|f\|_p \leq \|g\|_p$.

Exercise 6. For any $f, g \in L^1(X, \mu)$ and $a, b \in \mathbb{R}$, we have $\int (af + bg) \, \mathrm{d}\mu = a \int f \, \mathrm{d}\mu + b \int g \, \mathrm{d}\mu$.

Hint: Simple $\rightsquigarrow_{\mathrm{MCT}} L^+ \rightsquigarrow L^1$.

Exercise 7. Let $f, g \in L^1(X, \mu)$. If $f = g$ μ-a.e., then $\int f \, \mathrm{d}\mu = \int g \, \mathrm{d}\mu$. Hint: Consider $\int (f - g) \, \mathrm{d}\mu$.

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