SUMMER 2025 READING GROUP ON ERGODIC THEORY

Exercise Sheet 5 (Zhaoshen Zhai): Applications and Generalizations of Birkhoff's Pointwise Ergodic Theorem

Throughout, let (X, μ, T) be a measure-preserving dynamical system. The purpose of this exercise sheet is to give some applications and generalizations of the Pointwise Ergodic Theorem, which for convenience, we provide a sketch here. For $f \in L^1(X, \mu)$ and $n \in \mathbb{N}$, let $A_n^T f \coloneqq \frac{1}{n} \sum_{i<n} f \circ T^i$.

Theorem. *If T is ergodic, then for any $f \in L^1(X, \mu)$, we have $\lim_n A_n^T f =_\mu \int f \, \mathrm{d}\mu$.*

Proof sketch. Assume $\int f \, \mathrm{d}\mu = 0$ and recall that $l \coloneqq \limsup_n A_n^T f : X \to \mathbb{R}$ is T-invariant, so l is constant μ-a.e. by ergodicity, say at $l_0 \in \mathbb{R}$. Suppose that $f^* \coloneqq l_0/2 > 0$, so for each $x \in X$, there is a minimal $\eta(x) \in \mathbb{N}$ such that $A_{\eta(x)}^T f(x) \geq f^*$. We are done if there is a uniform $n \in \mathbb{N}$ such that $A_n^T f \geq f^*/2$. This is not true in general, but after trimming measure-ε parts of X, something like this can be done using:

> **Lemma** (Tiling Lemma). *Let $\eta : X \to \mathbb{N}$ be an arbitrary measurable function. For any $\varepsilon > 0$, there exists $n \gg 0$ such that for each $x \in X$ except on a measure-ε set, the interval $I_n^T(x)$ can be tiled, up to an ε-fraction, by intervals of the form $I_y \coloneqq I_{\eta(y)}^T(y)$ for $y \in X$.* ∎

Exercise 1. In the above context, prove that if both f and η are bounded, then $A_n^T f \geq f^*/2$.
Hint: Prove a stronger Tiling Lemma in this case.

Exercise 2. What is the average value of a given digit $0 \leq m \leq 9$, say $m \coloneqq 7$, to occur in the decimal expansion $0.x_1x_2\ldots$ of λ-a.e. $x \in [0, 1]$? That is, does $\lim_n \frac{1}{n}|\{i < n : x_i = m\}|$ exist, and what is it?
Hint: Consider the 10-ary Baker's map $b_{10} : [0, 1) \to [0, 1)$ sending $x \mapsto 10x \pmod 1$, which is isomorphic to the shift map on $10^{\mathbb{N}}$.

Exercise 3 (Equidistribution Theorem). A sequence $(x_n)_n$ in S^1 is said to be *equidistributed* if for every interval $I \subseteq S^1$, we have $\lim_n \frac{1}{n}|\{x_i\}_{i<n} \cap I| = \lambda(I)$. Prove that if $x_n = n\alpha$ for some irrational $\alpha \in S^1$, then $(x_n)_n$ is equidistributed. Hint: Don't overthink it.

Exercise 4 (Law of Large Numbers). If you know statistics, prove it! If not, skip it.

Exercise 5 (An ergodic theorem for non-ergodic actions). Intuitively, Birkhoff's Pointwise Ergodic Theorem states that ergodic transformations $T : X \to X$ stir up X so well that they spread any $f \in L^1(X, \mu)$ evenly on X, making it constant at $\int f \, \mathrm{d}\mu$; indeed, '$f \circ T^\infty = \int f \, \mathrm{d}\mu$'.

If T is not ergodic, then there is a non-trivial partition $X = X_1 \sqcup X_2$ into T-invariant pieces. The best that one can hope is at after 'enough' partitions $X = \bigsqcup_i X_i$, T still spreads each $f_i \coloneqq f\chi_{X_i}$ evenly on X_i. Viewing f from the lens of these T-invariant pieces leads to the *conditional expectation* of f:

> **Definition.** Let $\mathcal{A} \subseteq \mathcal{B}(X)$ be a sub-σ-algebra of $\mathcal{B}(X)$. For each $f \in L^1(X, \mu)$, there is a unique (up to a μ-null set) $\mathcal{A}$-measurable function $f_\mathcal{A}$ such that $\int_A f \, \mathrm{d}\mu = \int_A f_\mathcal{A} \, \mathrm{d}\mu$ for each $A \in \mathcal{A}$, called the *conditional expectation of f w.r.t. $\mathcal{A}$*. We write $\mathbb{E}(f|\mathcal{A})$ for $f_\mathcal{A}$.
>
> **Remark.** If $\mathcal{P} \subseteq \mathcal{B}(X)$ is a countable partition of X, then $\mathbb{E}(f|\langle\mathcal{P}\rangle_\sigma) = \sum_{P \in \mathcal{P}} \left(\frac{1}{\mu(P)} \int_P f \, \mathrm{d}\mu\right) \chi_P$.

Prove that for any (not necessarily ergodic) pmp transformation $T : X \to X$ and any $f \in L^1(X, \mu)$, we have $\lim_n A_n^T f =_\mu \mathbb{E}(f|\mathcal{B}_T)$, where $\mathcal{B}_T \subseteq \mathcal{B}(X)$ is the σ-algebra of all T-invariant Borel sets of X.
Hint: Same as the regular proof, only that $f^* : X \to \mathbb{R}$ is not necessarily constant, but just T-invariant.

Exercise 6 (L^p-ergodic theorem). Prove that for any $p \geq 1$, we have $A_n^T f \to_{L^p} \mathbb{E}(f|\mathcal{B}_T)$ for all $f \in L^p(X, \mu)$.
Hint: If f is bounded, then we are done by the DCT. Otherwise, let $f_k \to_{L^p} f$ where each f_k is bounded and triangle-inequality your way through, using that $\|A_n^T f\|_{L^p} \leq \|f\|_{L^p}$ (prove this too).

Date: May 28, 2025.