SUMMER 2025 READING GROUP ON ERGODIC THEORY

EXERCISE SHEET 6 (LUDOVIC RIVET): THE DENSITY OF SETS OF INTEGERS

Exercise 1. Compute the upper and lower densities of the following sets $A \subseteq \mathbb{N}$. Do they agree?

1. $A \coloneqq \{n \in \mathbb{N} : \forall m > 1(m^2 \nmid n)\}$, the square-free integers.
2. $A \coloneqq$ prime numbers. HINT: Szemerédi vs. Green-Tao.
3. $A \coloneqq$ numbers with an odd number of digits.

Exercise 2 (Furstenberg-Sárközy)**.** Let $A \subseteq \mathbb{N}$ be a subset with positive upper density and let $p \in \mathbb{Z}[x]$ be a polynomial with $p(0) = 0$. Using the recurrence theorem below, prove that there exists $a, b \in A$ and some $n \geq 1$ such that $x - y = p(n)$. HINT: Use the correspondence theorem.

> **Theorem 3** (Polynomial Recurrence)**.** *Let (X, μ, T) be a measure-preserving dynamical system. For any positive-measure $A \subseteq X$, there exists $n \geq 1$ such that $\mu(A \cap T^{-p(n)}A) > 0$.*

Date: June 10, 2025.